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S_8-MODULE STRUCTURES OF FREE ANTI-COMMUTATIVE ALGEBRA

Abstract. An algebra with identity $ab = -ba$ is called anti-commutative algebra. In this work we study S_8-module structures of free anti-commutative algebra of degree 8.

Keywords: anti-commutative algebra, symmetric functions, Schur functions, binary trees, S_n-module structures.

Introduction

An algebra A over a field K is called anti-commutative if for all elements a, b ∈ A, the following identity holds

$$ab = -ba. \quad (1)$$

Among the various papers, we consider as fundamental Kass and Witthoft's research for anti-commutative algebra of degree 4 with method of irreducible identities in 1970[4]. The key novelty of their study was using Osborn's method.[5] Similar works with aim to find S_n-module decomposition of free algebra can be seen in Yu. A. Bakhturin's [2] and C. Reutenauer's papers [6].
In [1] Bremner considered S_n-module structures of free anti-commutative algebra of degree $n \leq 7$. Our aim in this work is to investigate S_8-module structures of free anti-commutative algebra of degree 8. In our investigation we will use elements of the theory of symmetric functions such as the plethysm of Schur functions.

Statement of Main Result

Let P_n^k be a space generated by *k*th binary tree with *n* leaves. Let S^λ be a Specht module for partition λ ⊢ n. Let K be a field of characteristic zero. All algebras, vector spaces and modules we consider will be over field K.

Theorem. As S_8–module

$$\begin{aligned} P_8^{(1)} \cong\ & S^{(1,1,1,1,1,1,1,1)} \oplus 6 \cdot S^{(2,1,1,1,1,1,1)} \oplus 15 \cdot S^{(2,2,1,1,1,1)} \oplus 19 \cdot S^{(2,2,2,1,1)} \oplus 9 \cdot S^{(2,2,2,2)} \\ & \oplus 15 \cdot S^{(3,1,1,1,1,1)} \oplus 40 \cdot S^{(3,2,1,1,1)} \oplus 40 \cdot S^{(3,2,2,1)} \oplus 30 \cdot S^{(3,3,1,1)} \oplus 21 \cdot S^{(3,3,2)} \\ & \oplus 20 \cdot S^{(4,1,1,1,1)} \oplus 45 \cdot S^{(4,2,1,1)} \oplus 26 \cdot S^{(4,2,2)} \oplus 30 \cdot S^{(4,3,1)} \oplus 5 \cdot S^{(4,4)} \\ & \oplus 15 \cdot S^{(5,1,1,1)} \oplus 24 \cdot S^{(5,2,1)} \oplus 9 \cdot S^{(5,3)} \oplus 6 \cdot S^{(6,1,1)} \oplus 5 \cdot S^{(6,2)} \oplus S^{(7,1)} \end{aligned}$$

$$P_8^{(2)} \cong S^{(2,1,1,1,1,1,1)} \oplus 4 \cdot S^{(2,2,1,1,1,1)} \oplus 6 \cdot S^{(2,2,2,1,1)} \oplus 3 \cdot S^{(2,2,2,2)}$$
$$\oplus 4 \cdot S^{(3,1,1,1,1,1)} \oplus 12 \cdot S^{(3,2,1,1,1)} \oplus 12 \cdot S^{(3,2,2,1)} \oplus 8 \cdot S^{(3,3,1,1)} \oplus 5 \cdot S^{(3,3,2)}$$
$$\oplus 6 \cdot S^{(4,1,1,1,1)} \oplus 12 \cdot S^{(4,2,1,1)} \oplus 6 \cdot S^{(4,2,2)} \oplus 5 \cdot S^{(4,3,1)} \oplus 4 \cdot S^{(5,1,1,1)}$$
$$\oplus 4 \cdot S^{(5,2,1)} \oplus S^{(6,1,1)}$$

$$P_8^{(3)} \cong S^{(1,1,1,1,1,1,1,1)} \oplus 5 \cdot S^{(2,1,1,1,1,1,1)} \oplus 11 \cdot S^{(2,2,1,1,1,1)} \oplus 13 \cdot S^{(2,2,2,1,10)} \oplus 6 \cdot S^{(2,2,2,2)}$$
$$\oplus 10 \cdot S^{(3,1,1,1,1,1)} \oplus 24 \cdot S^{(3,2,1,1,1)} \oplus 23 \cdot S^{(3,2,2,1)} \oplus 16 \cdot S^{(3,3,1,1)} \oplus 11 \cdot S^{(3,3,2)}$$
$$\oplus 10 \cdot S^{(4,1,1,1,1)} \oplus 21 \cdot S^{(4,2,1,1)} \oplus 12 \cdot S^{(4,2,2)} \oplus 13 \cdot S^{(4,3,1)} \oplus 2 \cdot S^{(4,4)}$$
$$\oplus 5 \cdot S^{(5,1,1,1)} \oplus 8 \cdot S^{(5,2,1)} \oplus 3 \cdot S^{(5,3)} \oplus S^{(6,1,1)} \oplus S^{(6,2)}$$

$$P_8^{(4)} \cong S^{(2,1,1,1,1,1,1)} \oplus 3 \cdot S^{(2,2,1,1,1,1)} \oplus 4 \cdot S^{(2,2,2,1,1)} \oplus 2 \cdot S^{(2,2,2,2)}$$
$$\oplus 3 \cdot S^{(3,1,1,1,1,1)} \oplus 7 \cdot S^{(3,2,1,1,1)} \oplus 7 \cdot S^{(3,2,2,1)} \oplus 4 \cdot S^{(3,3,1,1)} \oplus 3 \cdot S^{(3,3,2)}$$
$$\oplus 3 \cdot S^{(4,1,1,1,1)} \oplus 5 \cdot S^{(4,2,1,1)} \oplus 3 \cdot S^{(4,2,2)} \oplus 2 \cdot S^{(4,3,1)} \oplus S^{(5,1,1,1)} \oplus S^{(5,2,1)}$$

$$P_8^{(5)} \cong S^{(1,1,1,1,1,1,1,1)} \oplus 5 \cdot S^{(2,1,1,1,1,1,1)} \oplus 11 \cdot S^{(2,2,1,1,1,1)} \oplus 13 \cdot S^{(2,2,2,1,1)} \oplus 6 \cdot S^{(2,2,2,2)}$$
$$\oplus 10 \cdot S^{(3,1,1,1,1,1)} \oplus 24 \cdot S^{(3,2,1,1,1)} \oplus 23 \cdot S^{(3,2,2,1)} \oplus 16 \cdot S^{(3,3,1,1)} \oplus 11 \cdot S^{(3,3,2)}$$
$$\oplus 10 \cdot S^{(4,1,1,1,1)} \oplus 21 \cdot S^{(4,2,1,1)} \oplus 12 \cdot S^{(4,2,2)} \oplus 13 \cdot S^{(4,3,1)} \oplus 2 \cdot S^{(4,4)}$$
$$\oplus 5 \cdot S^{(5,1,1,1)} \oplus 8 \cdot S^{(5,2,1)} \oplus 3 \cdot S^{(5,3)} \oplus S^{(6,1,1)} \oplus S^{(6,2)}$$

$$P_8^{(6)} \cong S^{(1,1,1,1,1,1,1,1)} \oplus 3 \cdot S^{(2,1,1,1,1,1,1)} \oplus 6 \cdot S^{(2,2,1,1,1,1)} \oplus 7 \cdot S^{(2,2,2,1,1)} \oplus 3 \cdot S^{(2,2,2,2)}$$
$$\oplus 4 \cdot S^{(3,1,1,1,1,1)} \oplus 12 \cdot S^{(3,2,1,1,1)} \oplus 11 \cdot S^{(3,2,2,1)} \oplus 9 \cdot S^{(3,3,1,1)} \oplus 5 \cdot S^{(3,3,2)}$$
$$\oplus 4 \cdot S^{(4,1,1,1,1)} \oplus 11 \cdot S^{(4,2,1,1)} \oplus 5 \cdot S^{(4,2,2)} \oplus 7 \cdot S^{(4,3,1)} \oplus S^{(4,4)}$$
$$\oplus 3 \cdot S^{(5,1,1,1)} \oplus 4 \cdot S^{(5,2,1)} \oplus S^{(5,3)} \oplus S^{(6,1,1)}$$

$$P_8^{(7)} \cong S^{(1,1,1,1,1,1,1,1)} \oplus 5 \cdot S^{(2,1,1,1,1,1,1)} \oplus 11 \cdot S^{(2,2,1,1,1,1)} \oplus 13 \cdot S^{(2,2,2,1,1)} \oplus 6 \cdot S^{(2,2,2,2)}$$
$$\oplus 10 \cdot S^{(3,1,1,1,1,1)} \oplus 24 \cdot S^{(3,2,1,1,1)} \oplus 23 \cdot S^{(3,2,2,1)} \oplus 16 \cdot S^{(3,3,1,1)} \oplus 11 \cdot S^{(3,3,2)}$$
$$\oplus 10 \cdot S^{(4,1,1,1,1)} \oplus 21 \cdot S^{(4,2,1,1)} \oplus 12 \cdot S^{(4,2,2)} \oplus 13 \cdot S^{(4,3,1)} \oplus 2 \cdot S^{(4,4)}$$
$$\oplus 5 \cdot S^{(5,1,1,1)} \oplus 8 \cdot S^{(5,2,1)} \oplus 3 \cdot S_8^{(5,3)} \oplus S^{(6,1,1)} \oplus S^{(6,2)}$$

$$P_8^{(8)} \cong S^{(2,1,1,1,1,1,1)} \oplus 3 \cdot S^{(2,2,1,1,1,1)} \oplus 4 \cdot S^{(2,2,2,1,1)} \oplus 2 \cdot S^{(2,2,2,2)}$$
$$\oplus 3 \cdot S^{(3,1,1,1,1,1)} \oplus 7 \cdot S^{(3,2,1,1,1)} \oplus 7 \cdot S^{(3,2,2,1)} \oplus 4 \cdot S^{(3,3,1,1)} \oplus 3 \cdot S^{(3,3,2)}$$
$$\oplus 3 \cdot S^{(4,1,1,1,1)} \oplus 5 \cdot S^{(4,2,1,1)} \oplus 3 \cdot S^{(4,2,2)} \oplus 2 \cdot S^{(4,3,1)} \oplus S^{(5,1,1,1)} \oplus S^{(5,2,1)}$$

$$P_8^{(9)} \cong S^{(1,1,1,1,1,1,1,1)} \oplus 4 \cdot S^{(2,1,1,1,1,1,1)} \oplus 8 \cdot S^{(2,2,1,1,1,1)} \oplus 9 \cdot S^{(2,2,2,1,1)} \oplus 4 \cdot S^{(2,2,2,2)}$$
$$\oplus 6 \cdot S^{(3,1,1,1,1,1)} \oplus 14 \cdot S^{(3,2,1,1,1)} \oplus 13 \cdot S^{(3,2,2,1)} \oplus 9 \cdot S^{(3,3,1,1)} \oplus 6 \cdot S^{(3,3,2)}$$
$$\oplus 4 \cdot S^{(4,1,1,1,1)} \oplus 10 \cdot S^{(4,2,1,1)} \oplus 5 \cdot S^{(4,2,2)} \oplus 7 \cdot S^{(4,3,1)} \oplus S^{(4,4)}$$
$$\oplus 4 \cdot S^{(5,1,1,1)} \oplus 5 \cdot S^{(5,2,1)} \oplus S^{(5,3)} \oplus S^{(6,1,1)}$$

$$P_8^{(10)} \cong S^{(1,1,1,1,1,1,1,1)} \oplus 5\cdot S^{(2,1,1,1,1,1,1)} \oplus 11\cdot S^{(2,2,1,1,1,1)} \oplus 13\cdot S^{(2,2,2,1,1)} \oplus 6\cdot S^{(2,2,2,2)}$$
$$\oplus 10\cdot S^{(3,1,1,1,1,1)} \oplus 24\cdot S^{(3,2,1,1,1)} \oplus 23\cdot S^{(3,2,2,1)} \oplus 16\cdot S^{(3,3,1,1)} \oplus 11\cdot S^{(3,3,2)}$$
$$\oplus 10\cdot S^{(4,1,1,1,1)} \oplus 21\cdot S^{(4,2,1,1)} \oplus 12\cdot S^{(4,2,2)} \oplus 13\cdot S^{(4,3,1)} \oplus 2\cdot S^{(4,4}$$
$$\oplus 5\cdot S^{(5,1,1,1)} \oplus 8\cdot S^{(5,2,1)} \oplus 3\cdot S^{(5,3)} \oplus S^{(6,1,1)} \oplus S^{(6,2)}$$

$$P_8^{(11)} \cong S^{(2,1,1,1,1,1,1)} \oplus 3\cdot S^{(2,2,1,1,1,1)} \oplus 4\cdot S^{(2,2,2,1,1)} \oplus 2\cdot S^{(2,2,2,2)}$$
$$\oplus 3\cdot S^{(3,1,1,1,1,1)} \oplus 7\cdot S^{(3,2,1,1,1)} \oplus 7\cdot S^{(3,2,2,1)} \oplus 4\cdot S^{(3,3,1,1)} \oplus 3\cdot S^{(3,3,2)}$$
$$\oplus 3\cdot S^{(4,1,1,1,1)} \oplus 5\cdot S^{(4,2,1,1)} \oplus 3\cdot S^{(4,2,2)} \oplus 2\cdot S^{(4,3,1)} \oplus S^{(5,1,1,1)} \oplus S^{(5,2,1)}$$

$$P_8^{(12)} \cong S^{(1,1,1,1,1,1,1,1)} \oplus 5\cdot S^{(2,1,1,1,1,1,1)} \oplus 11\cdot S^{(2,2,1,1,1,1)} \oplus 13\cdot S^{(2,2,2,1,1)} \oplus 6\cdot S^{(2,2,2,2)}$$
$$\oplus 10\cdot S^{(3,1,1,1,1,1)} \oplus 24\cdot S^{(3,2,1,1,1)} \oplus 23\cdot S^{(3,2,2,1)} \oplus 16\cdot S^{(3,3,1,1)} \oplus 11\cdot S^{(3,3,2)}$$
$$\oplus 10\cdot S^{(4,1,1,1,1)} \oplus 21\cdot S^{(4,2,1,1)} \oplus 12\cdot S^{(4,2,2)} \oplus 13\cdot S^{(4,3,1)} \oplus 2\cdot S^{(4,4)}$$
$$\oplus 5\cdot S^{(5,1,1,1)} \oplus 8\cdot S^{(5,2,1)} \oplus 3\cdot S^{(5,3)} \oplus S^{(6,1,1)} \oplus S^{(6,2)}$$

$$P_8^{(13)} \cong S^{(2,1,1,1,1,1,1)} \oplus 3\cdot S^{(2,2,1,1,1,1)} \oplus 4\cdot S^{(2,2,2,1,1)} \oplus 2\cdot S^{(2,2,2,2)}$$
$$\oplus 3\cdot S^{(3,1,1,1,1,1)} \oplus 7\cdot S^{(3,2,1,1,1)} \oplus 7\cdot S^{(3,2,2,1)} \oplus 4\cdot S^{(3,3,1,1)} \oplus 3\cdot S^{(3,3,2)}$$
$$\oplus 3\cdot S^{(4,1,1,1,1)} \oplus 5\cdot S^{(4,2,1,1)} \oplus 3\cdot S^{(4,2,2)} \oplus 2\cdot S^{(4,3,1)} \oplus S^{(5,1,1,1)} \oplus S^{(5,2,1)}$$

$$P_8^{(14)} \cong S^{(1,1,1,1,1,1,1,1)} \oplus 4\cdot S^{(2,1,1,1,1,1,1)} \oplus 8\cdot S^{(2,2,1,1,1,1)} \oplus 9\cdot S^{(2,2,2,1,1)} \oplus 4\cdot S^{(2,2,2,2)}$$
$$\oplus 6\cdot S^{(3,1,1,1,1,1)} \oplus 14\cdot S^{(3,2,1,1,1)} \oplus 13\cdot S^{(3,2,2,1)} \oplus 9\cdot S^{(3,3,1,1)} \oplus 6\cdot S^{(3,3,2)}$$
$$\oplus 4\cdot S^{(4,1,1,1,1)} \oplus 9\cdot S^{(4,2,1,1)} \oplus 5\cdot S^{(4,2,2)} \oplus 6\cdot S^{(4,3,1)} \oplus S^{(4,4)}$$
$$\oplus 4\cdot S^{(5,1,1,1)} \oplus 5\cdot S^{(5,2,1)} \oplus S^{(5,3)} \oplus S^{(6,1,1)}$$

$$P_8^{(15)} \cong S^{(2,1,1,1,1,1,1)} \oplus 2\cdot S^{(2,2,1,1,1,1)} \oplus 3\cdot S^{(2,2,2,1,1)} \oplus S^{(2,2,2,2)}$$
$$\oplus 2\cdot S^{(3,1,1,1,1,1)} \oplus 4\cdot S^{(3,2,1,1,1)} \oplus 4\cdot S^{(3,2,2,1)} \oplus 2\cdot S^{(3,3,1,1)} \oplus 2\cdot S^{(3,3,2)}$$
$$\oplus S^{(4,1,1,1,1)} \oplus 2\cdot S^{(4,2,1,1)} \oplus S^{(4,2,2)} \oplus S^{(4,3,1)}$$

$$P_8^{(16)} \cong S^{(1,1,1,1,1,1,1,1)} \oplus 5\cdot S^{(2,1,1,1,1,1,1)} \oplus 11\cdot S^{(2,2,1,1,1,1)} \oplus 13\cdot S^{(2,2,2,1,1)} \oplus 6\cdot S^{(2,2,2,2)}$$
$$\oplus 10\cdot S^{(3,1,1,1,1,1)} \oplus 24\cdot S^{(3,2,1,1,1)} \oplus 23\cdot S^{(3,2,2,1)} \oplus 16\cdot S^{(3,3,1,1)} \oplus 11\cdot S^{(3,3,2)}$$
$$\oplus 10\cdot S^{(4,1,1,1,1)} \oplus 21\cdot S^{(4,2,1,1)} \oplus 12\cdot S^{(4,2,2)} \oplus 13\cdot S^{(4,3,1)} \oplus 2\cdot S^{(4,4)}$$
$$\oplus 5\cdot S^{(5,1,1,1)} \oplus 8\cdot S^{(5,2,1)} \oplus 3\cdot S^{(5,3)} \oplus S^{(6,1,1)} \oplus S^{(6,2)}$$

$$P_8^{(17)} \cong S^{(1,1,1,1,1,1,1,1)} \oplus 2\cdot S^{(2,1,1,1,1,1,1)} \oplus 5\cdot S^{(2,2,1,1,1,1)} \oplus 4\cdot S^{(2,2,2,1,1)} \oplus 3\cdot S^{(2,2,2,2)}$$
$$\oplus 2\cdot S^{(3,1,1,1,1,1)} \oplus 7\cdot S^{(3,2,1,1,1)} \oplus 6\cdot S^{(3,2,2,1)} \oplus 6\cdot S^{(3,3,1,1)} \oplus 2\cdot S^{(3,3,2)}$$
$$\oplus 2\cdot S_8^{(4,1,1,1,1)} \oplus 4\cdot S^{(4,2,1,1)} \oplus 3\cdot S^{(4,2,2)} \oplus 3\cdot S^{(4,3,1)} \oplus S^{(4,4)}$$
$$\oplus S^{(5,1,1,1)} \oplus S^{(5,2,1)}$$

$$P_8^{(18)} \cong S^{(1,1,1,1,1,1,1,1)} \oplus 5 \cdot S^{(2,1,1,1,1,1,1)} \oplus 11 \cdot S^{(2,2,1,1,1,1)} \oplus 13 \cdot S^{(2,2,2,1,1)} \oplus 6 \cdot S^{(2,2,2,2)}$$
$$\oplus 10 \cdot S^{(3,1,1,1,1,1)} \oplus 24 \cdot S^{(3,2,1,1,1)} \oplus 23 \cdot S^{(3,2,2,1)} \oplus 16 \cdot S^{(3,3,1,1)} \oplus 11 \cdot S^{(3,3,2)}$$
$$\oplus 10 \cdot S^{(4,1,1,1,1)} \oplus 21 \cdot S^{(4,2,1,1)} \oplus 12 \cdot S^{(4,2,2)} \oplus 13 \cdot S^{(4,3,1)} \oplus 2 \cdot S^{(4,4)}$$
$$\oplus 5 \cdot S^{(5,1,1,1)} \oplus 8 \cdot S^{(5,2,1)} \oplus 3 \cdot S^{(5,3)} \oplus S_8^{(6,1,1)} \oplus S^{(6,2)}$$

$$P_8^{(19)} \cong S^{(2,1,1,1,1,1,1)} \oplus 3 \cdot S^{(2,2,1,1,1,1)} \oplus 4 \cdot S^{(2,2,2,1,1)} \oplus 2 \cdot S^{(2,2,2,2)}$$
$$\oplus 3 \cdot S^{(3,1,1,1,1,1)} \oplus 7 \cdot S^{(3,2,1,1,1)} \oplus 7 \cdot S^{(3,2,2,1)} \oplus 4 \cdot S^{(3,3,1,1)} \oplus 3 \cdot S^{(3,3,2)}$$
$$\oplus 3 \cdot S^{(4,1,1,1,1)} \oplus 5 \cdot S^{(4,2,1,1)} \oplus 3 \cdot S^{(4,2,2)} \oplus 2 \cdot S^{(4,3,1)} \oplus S_8^{(5,1,1,1)} \oplus S^{(5,2,1)}$$

$$P_8^{(20)} \cong S^{(1,1,1,1,1,1,1,1)} \oplus 5 \cdot S^{(2,1,1,1,1,1,1)} \oplus 11 \cdot S^{(2,2,1,1,1,1)} \oplus 13 \cdot S^{(2,2,2,1,1)} \oplus 6 \cdot S^{(2,2,2,2)}$$
$$\oplus 10 \cdot S^{(3,1,1,1,1,1)} \oplus 24 \cdot S^{(3,2,1,1,1)} \oplus 23 \cdot S^{(3,2,2,1)} \oplus 16 \cdot S^{(3,3,1,1)} \oplus 11 \cdot S^{(3,3,2)}$$
$$\oplus 10 \cdot S^{(4,1,1,1,1)} \oplus 21 \cdot S^{(4,2,1,1)} \oplus 12 \cdot S^{(4,2,2)} \oplus 13 \cdot S^{(4,3,1)} \oplus 2 \cdot S^{(4,4)}$$
$$\oplus 5 \cdot S^{(5,1,1,1)} \oplus 8 \cdot S^{(5,2,1)} \oplus 3 \cdot S^{(5,3)} \oplus S^{(6,1,1)} \oplus S^{(6,2)}$$

$$P_8^{(21)} \cong S^{(2,1,1,1,1,1,1)} \oplus 3 \cdot S^{(2,2,1,1,1,1)} \oplus 4 \cdot S^{(2,2,2,1,1)} \oplus 2 \cdot S^{(2,2,2,2)}$$
$$\oplus 3 \cdot S^{(3,1,1,1,1,1)} \oplus 7 \cdot S^{(3,2,1,1,1)} \oplus 7 \cdot S^{(3,2,2,1)} \oplus 4 \cdot S^{(3,3,1,1)} \oplus 3 \cdot S^{(3,3,2)}$$
$$\oplus 3 \cdot S^{(4,1,1,1,1)} \oplus 5 \cdot S^{(4,2,1,1)} \oplus 3 \cdot S^{(4,2,2)} \oplus 2 \cdot S^{(4,3,1)} \oplus S^{(5,1,1,1)} \oplus S^{(5,2,1)}$$

$$P_8^{(22)} \cong S^{(2,2,2,1,1)} \oplus S^{(3,1,1,1,1,1)} \oplus S^{(3,2,1,1,1)} \oplus S^{(3,2,2,1)} \oplus S^{(3,3,2)} \oplus S^{(4,2,1,1)}.$$

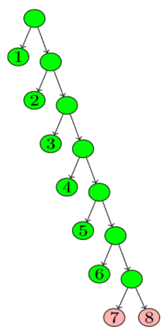


Figure 1: Binary Tree 1

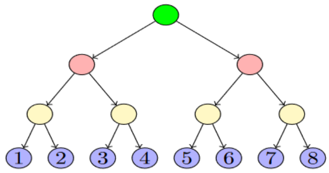
We will consider proof for Tree 1 and 22. First, let's label each leaf with numbers {1, 2, . . . , 8} from bottom to top. See Fig. 1

Obviously, the group of symmetries of the binary tree 1 is $Sym(T_1) = \langle(12)\rangle$. Moreover, the binary tree 1 corresponds to the following non-associative monomial

$$T_1 := ((((((12)3)4)5)6)7)8.$$

Now, we label each leaf of T_{22} by numbers {1, 2, … , 8} in the following way

Figure 2: Binary Tree 22



Lemma 1. Let T_1 be the binary tree 1 and let σ ∈ S_2 be a permutation. Then

$$\sigma T_1 = T_1, \text{ if } \sigma = (1),$$
$$\sigma T_1 = -T_1, \text{ if } \sigma = (12).$$

Proof.
If σ = (1), then it is clear. If σ = (12), then

$$(12)\circ(((((((12)3)4)5)6)7)8) = (((((((21)3)4)5)6)7)8) = (1) = (((((((12)3)4)5)6)7)8).$$

□

Lemma 2. *Let* S_2 *be the symmetric group on set* {1, 2}. *Then*

$$Sym(T_{22}) \cong S_2 \wr (S_2 \wr S_2).$$

Proof. Set

$$a = (12),\ b = (13)(24),\ c = (15)(26)(37)(48).$$

Define bijective function f: $Sym(T_{22}) \to S_2 \wr (S_2 \wr S_2)$ in the following way

$$f(a) = \{[(12), (1); (1)], [(1), (1); (1)]; (1)\},$$

$$f(b) = \{[(1), (1); (12)], [(1), (1); (1)]; (1)\},$$

$$f(c) = \{[(1), (1); (1)], [(1), (1); (1)]; (12)\}.$$

Now let us show that the function f is a group morphism. Applying the function f to $a \circ b$, $b \circ a$, $a \circ c$, $c \circ a$, $b \circ c$ *and* $c \circ b$ we get

$$f(a \circ b) = f(1324) = \{[(1), (12); (12)], [(1), (1); (1)]; (1)\};$$
$$f(b \circ a) = f(1423) = \{[(1), (12); (12)], [(1), (1); (1)]; (1)\};$$
$$f(a \circ c) = f((1526)(37)(48)) = \{[(1), (1); (1)], [(12), (1); (1)]; (12)\};$$
$$f(c \circ a) = f((1625)(37)(48)) = \{[(12), (1); (1)], [(1), (1); (1)]; (12)\};$$
$$f(b \circ c) = f((1537)(2648)) = \{[(1), (1); (12)], [(1), (1); (1)]; (12)\};$$
$$f(c \circ b) = f((1735)(4628)) = \{[(1), (1); (1)], [(12), (12); (12)]; (12)\}.$$

Now let's compute+

$$f(a)\cdot f(b) = \{[(12),(1);(1)],[(1),(1);(1)];(1)\}\cdot\{[(1),(1);(12)],[(1),(1);(1)];(1)\}$$
$$= \{[(1),(12);(12)],[(1),(1);(1)];(1)\}.$$

$$f(b)\cdot f(a) = \{[(1),(1);(12)],[(1),(1);(1)];(1)\}\cdot\{[(12),(1);(1)],[(1),(1);(1)];(1)\}$$
$$= \{[(1),(12);(12)],[(1),(1);(1)];(1)\}$$

$$f(a)\cdot f(c) = \{[(12),(1);(1)],[(1),(1);(1)];(1)\}\cdot\{[(1),(1);(1)],[(1),(1);(1)];(12)\}$$
$$= \{[(1),(1);(1)],[(12),(1);(1)];(12)\}$$

$$f(c)\cdot f(a) = \{[(1),(1);(1)],[(1),(1);(12)];(1)\}\cdot\{[(12),(1);(1)],[(1),(1);(1)];(1)\}$$
$$= \{[(12),(1);(1)],[(1),(1);(1)];(12)\}$$

$$f(b)\cdot f(c) = \{[(1),(1);(12)],[(1),(1);(1)];(1)\}\cdot\{[(1),(1);(1)],[(1),(1);(1)];(12)\}$$
$$= \{[(1),(1);(12)],[(1),(1);(1)];(12)\}$$

$$f(c)\cdot f(b) = \{[(1),(1);(1)],[(1),(1);(1)];(12)\}\cdot\{[(1),(1);(12)],[(1),(1);(1)];(1)\}$$
$$= \{[(1),(1);(1)],[(12),(12);(12)];(12)\}$$

Hence

$$f(a\circ b) = f(a)\cdot f(b);$$
$$f(b\circ a) = f(b)\cdot f(a);$$
$$f(a\circ c) = f(a)\cdot f(c);$$
$$f(c\circ a) = f(c)\cdot f(a);$$
$$f(b\circ c) = f(b)\cdot f(c);$$
$$f(c\circ b) = f(c)\cdot f(b).$$

□

Lemma 3. *Let T_{22} be the binary tree and let $\sigma \in S_2$ be a permutation. Then*

$$\sigma T_{22} = T_{22},\ \textit{if}\ \sigma = (1);$$
$$\sigma T_{22} = -T_{22},\ \textit{if}\ \sigma = (12);$$
$$\sigma T_{22} = -T_{22},\ \textit{if}\ \sigma = (13)(24);$$
$$\sigma T_{22} = -T_{22},\ \textit{if}\ \sigma = (15)(26)(37)(48).$$

Proof. if $\sigma = (1)$, then it is obvious. If $\sigma = (12)$, then

$$(12)\circ((12)(34))((56)(78)) = ((21)(34))((56)(78)) = (1) = -((12)(34))((56)(78))$$

If $\sigma = (13)(24)$, then

$$(13)(24)\circ((12)(34))((56)(78)) = ((34)(12))((56)(78)) = (1) = -((12)(34))((56)(78))$$

If $\sigma = (15)(26)(37)(48)$, then

$$(15)(26)(37)(48) \circ ((12)(34))((56)(78)) = ((56)(78))((12)(34)) = (1) = -((12)(34))((56)(78))$$

□

Proof of Theorem. Using Lemma 1 we get

$$P_8^1 \cong Ind_{S_1\times S_1\times S_1\times S_1\times S_1\times S_1\times S_2}^{S_8}(\mathbf{1}_{S_1}^- \otimes \mathbf{1}_{S_1}^- \otimes \mathbf{1}_{S_1}^- \otimes \mathbf{1}_{S_1}^- \otimes \mathbf{1}_{S_1}^- \otimes \mathbf{1}_{S_1}^- \otimes \mathbf{1}_{S_2}^-) \cong$$

$$S^{(1,1,1,1,1,1,1,1)} \oplus 6\cdot S^{(2,1,1,1,1,1,1)} \oplus 15\cdot S^{(2,2,1,1,1,1)} \oplus 19\cdot S^{(2,2,2,1,1)} \oplus 9\cdot S^{(2,2,2,2)}$$
$$\oplus 15\cdot S^{(3,1,1,1,1,1)} \oplus 40\cdot S^{(3,2,1,1,1)} \oplus 40\cdot S^{(3,2,2,1)} \oplus 30\cdot S^{(3,3,1,1)} \oplus 21\cdot S^{(3,3,2)}$$
$$\oplus 20\cdot S^{(4,1,1,1,1)} \oplus 45\cdot S^{(4,2,1,1)} \oplus 26\cdot S^{(4,2,2)} \oplus 30\cdot S^{(4,3,1)} \oplus 5\cdot S^{(4,4)}$$
$$\oplus 15\cdot S^{(5,1,1,1)} \oplus 24\cdot S^{(5,2,1)} \oplus 9\cdot S^{(5,3)} \oplus 6\cdot S^{(6,1,1)} \oplus 5\cdot S^{(6,2)} \oplus S^{(7,1)}.$$

Using *Lemma 2* and *Lemma 3* we get

$$P_8^{22} \cong Ind_{S_2\wr(S_2\wr S_2)}^{s_8}(\mathbf{1}_{S_2}^- \otimes (\mathbf{1}_{S_2}^- \otimes \mathbf{1}_{S_2}^-)).$$

To calculate this induced representation, we need the theory of symmetric functions. We apply characteristic map on the induced representation above we get the following plethysm of Schur functions

$$Ch(Ind_{S_2\wr(S_2\wr S_2)}^{s_8}(\mathbf{1}_{S_2}^- \otimes (\mathbf{1}_{S_2}^- \otimes \mathbf{1}_{S_2}^-)) =$$
$$= s_{(1,1)} \circ (s_{(1,1)} \circ s_{(1,1)}).$$

Calculating the plethysm above we get

$$s_{(1,1)} \circ (s_{(1,1)} \circ s_{(1,1)}) = s_{(2,2,2,1,1)} + s_{(3,1,1,1,1,1)} + s_{(3,2,2,1)} + s_{(4,2,1,1)}$$

□

The rest $P_8^2 - P_8^{21}$ are proven in a similar way.

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ЕРКІН АНТИ-КОММУТАТИВТІ АЛГЕБРАНЫҢ S_8-МОДУЛЬ ҚҰРЫЛЫМДАРЫ

Андатпа. *ab* = - *ba* теңдігін қанағаттандыратын алгебра анти-коммутативті алгебра деп аталады. Бұл жұмыста біз дәрежесі 8 анти-коммутативті алгебраның симметриялық құрылымдарын зерттейміз.

Түйін-сөздер: анти-коммутативті алгебра, симметриялы функциялар, Шур функциялары, бинарлы ағаштар S_n-модульдік құрылымдар.

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S_8-МОДУЛЬНЫЕ СТРУКТУРЫ СВОБОДНОЙ АНТИ-КОММУТАТИВНОЙ АЛГЕБРЫ

Андатпа. Алгебра с тождеством *ab* = −*ba* называется анти-коммутативной алгеброй. В данной работе мы изучаем симметричные модульные структуры свободной анти-коммутативной алгебры степени 8.

Ключевые слова: анти-коммутативная алгебра, симметричные функции, функции Шура, бинарные деревья, S_n-модульные структуры.

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