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TYPES OF LYAPUNOV FUNCTIONS AND GLOBAL STABILITY

Abstract. The stability analysis of dynamical systems is a critical aspect of understanding their behavior and ensuring reliable performance in various applications. Lyapunov functions play a pivotal role in this analysis, offering a powerful mathematical tool to assess the stability of equilibrium points. Despite the fact that at the moment there is no general method for finding the Lyapunov function, this review explores the diverse landscape of Lyapunov functions and their contributions to achieving global stability in dynamical systems. Examples of epidemic models and their Lyapunov functions with the graphical implementations in Python in different reproduction numbers are included for deeper understanding. In addition, the history of the Lyapunov global asymptotic stability theorem was considered shortly. Possible further research ideas in this area are included at the end.

Keywords: Lyapunov functions, global stability, dynamical systems, SIR, SEIR, stability.

Introduction

Lyapunov stability theory was developed by Aleksandr Lyapunov, a Russian mathematician in 1892, and came from his doctoral dissertation. Until now, the theory of Lyapunov stability is still important to the stability theory of dynamical systems.[4] Knowing whether the dynamical system is globally stable or not is very important, because for example we can decide is epidemic dangerous or not. This allows engineers, mathematicians, and scientists to design reliable and efficient control strategies.

Definition 1: A function $L: \mathbb{R}^n \to \mathbb{R}$ is positive definite (PD) if

- $L(z) \geq 0$ for all z
- $L(z) = 0$ if and only if $z = 0$
- all sublevel sets of L are bounded

last condition equivalent to $L(z) \to \infty$ as $z \to \infty$.

Theorem 1: Suppose there is a function L such that

- L is positive definite
- $\dot{L}(z) < 0$ for all $z \neq 0$, $\dot{L}(z) = 0$

then, every trajectory of $\dot{x} = f(x)$ converges to equilibrium point as $t \to \infty$ (i.e., the system is globally asymptotically stable). Where z = x(t); $\dot{x} = f(x)$. [5]

In other words, according to the theorem, if a Lyapunov function can be identified that is positive definite, radially unbounded, and exhibits a negative definite time derivative along system trajectories, then the equilibrium point under consideration is globally asymptotically stable. This theorem 1 is called the Lyapunov global asymptotic stability theorem. Finding the Lyapunov function is the hardest part of proving the global stability of epidemic models using Theorem 1. Because there is no general method to do it. Despite this fact, we collected some known types of Lyapunov functions in the next chapter.

Types of Lyapunov functions

1. The logarithmic Lyapunov function proposed by Goh for Lotka-Volterra systems:

$$L(x_1, x_2, \dots, x_n) = \Sigma_{i=1}^{n} c_i \left(x_i - x_i^* - x_i^* ln \frac{x_i}{x_i^*} \right)$$

This function is derived from the primary integral of the well-known Lotka-Volterra prey-predator system and gained prominence through a series of publications by Bean-San Goh [10, 11, 12].

2. Common quadratic Lyapunov function, have been extensively exploited for nonlinear and linear systems:

$$V(x_1, x_2, \dots, x_n) = \Sigma_{i=1}^{n} \frac{c_i}{2} (x_i - x_i^*)^2$$

3. Composite quadratic function:

$$W(x_1, x_2, \dots, x_n) = \frac{c}{2} [\Sigma_{i=1}^{n} (x_i - x_i^*)]^2$$

4. Linear combination of infectious compartments:

$$L(x_1, x_2, \dots, x_n) = \Sigma_{i \geq 2}^{n} c_i x_i$$

5. Lyapunov functions are formulated as integrals encompassing the dynamics of the model. Typically, the integration interval initiates at a designated Endemic Equilibrium (EE) value x_i^* and concludes at the same x_i. This approach proves advantageous when ensuring the uniqueness of the EE is guaranteed, even though determining the precise values of the EE may pose analytical challenges.

Integral Lyapunov functions prove particularly advantageous when the model involves multiple stages of infection, leading to a transition from an exponential distribution of the infectious period to a gamma distribution [1, 2, 6, 7, 8, 9]. These integral Lyapunov functions, albeit taking different forms, find widespread application in models featuring explicit delays, such as systems of Delay Differential Equations, and age-structured models.

Epidemic models

In this chapter, we use the above-mentioned types of Lyapunov functions to prove the global stability of SIS, SIR, and SIRS epidemic models.

SIS model

Certain diseases do not provide enduring immunity, leaving individuals susceptible to reinfection. These infections lack a recovered state, and individuals may become susceptible again after being infected. A suitable model for such diseases is the SIS type. In this model, the total population N is divided into two compartments, where N equals the sum of individuals in the susceptible class (S) and those who are infectious (I). The SIS model depicts a recurring cycle wherein individuals move from the susceptible state (S) to the infectious state (I) and back to the susceptible state (S), as illustrated in Figure 1.

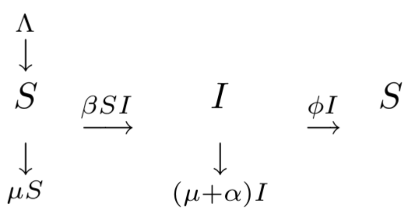

Figure 1: SIS model

The system of equations of model SIS [3]:

$$\begin{cases} \dfrac{dS}{dt} = \Lambda - \beta SI - \mu S + \phi I \\ \dfrac{dI}{dt} = \beta SI - (\phi + \mu + \alpha)I \end{cases}$$

The parameters in the model are positive constants. The constant Λ represents the recruitment rate of susceptibles, accounting for births and immigration, while μ denotes the per capita natural mortality rate. In our assumptions, a disease has the potential to be fatal for some infected individuals; therefore, we incorporate deaths due to the disease in the model using the disease-related death rate from the infectious class, denoted as α. Additionally, φ signifies the rate at which individuals move from the infectious class back to the susceptible class, without acquiring immunity.

The disease-free equilibrium (DFE) $E^0 = (S^0, I^0) = \left(\frac{\Lambda}{\mu}, 0\right)$ and basic reproduction number $\mathcal{R}_0 = \frac{\Lambda\beta}{\mu(\phi+\mu+\alpha)}$. Basic reproductive number was found by Next generation matrix method [13]. The term $\mathcal{R}_0$ is characterized as the mean count of secondary infections that arise when a single infectious individual is introduced to a host population that is entirely susceptible. Feasible region:

$G = \left\{(S, I) \in \mathbb{R}^2{}_+ :\ S \geq 0,\ \ I \geq 0,\ S + I \leq \frac{\Lambda}{\mu}\right\}$ which is positively invariant with respect to SIS model's system of equations.

Theorem 2: The DFE E^0 is globally asymptotically stable in G if $\mathcal{R}_0 \leq 1$.

Proof: To prove it we can define $L: \{(S, I) \in G: S, I > 0\}$:

$$L(S, I) = \frac{1}{2}[(S - S^0) + I]^2 + \frac{\alpha + 2\mu}{\beta} I$$

Obviously, $L > 0$ and $L(S_0, I_0) = 0$. The time derivative of L is not positive if $\mathcal{R}_0 \leq 1$:

$$\frac{dL}{dt} = -\mu(S - S^0)^2 - (\mu + \alpha)I^2 - \frac{(\alpha + 2\mu)(\phi + \mu + \alpha)}{\beta}(1 - \mathcal{R}_0)I$$

Therefore, the DFE E^0 is globally asymptotically stable in G if $\mathcal{R}_0 \leq 1$.

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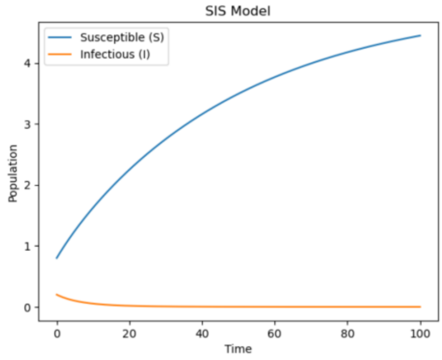

Figure 2: SIS model implementation in Python when $\mathcal{R}_0 \leq 1$

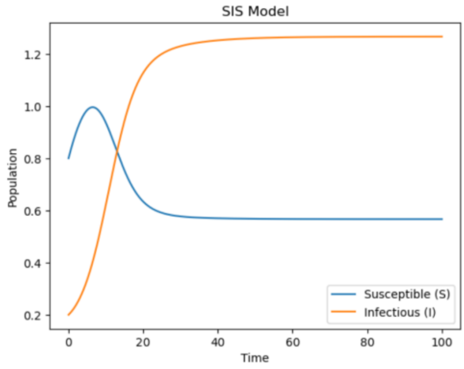

Figure 3: SIS model implementation in Python when $\mathcal{R}_0 > 1$

Let's consider the endemic equilibrium (EE) of the SIS model:

$$\mathrm{E}^* = (S^*, I^*) = \left(\frac{S^0}{R^0}, \frac{\mu(\phi + \mu + \alpha)}{\beta(\alpha + 2\mu)}(\mathcal{R}_0 - 1)\right).$$

Theorem 3: The EE E^* is globally asymptotically stable in G if $\mathcal{R}_0 > 1$.

Proof: To prove it we can define function using combination of composite quadratic Lyapunov function and logarithmic Lyapunov function,

$$L: \{(S, I) \in G: S, I > 0\}$$

$$L(S, I) = \frac{1}{2}[(S - S^*) + (I - I^*)]^2 + \frac{\alpha + 2\mu}{\beta}\left(I - I^* - I^* ln\frac{I}{I^*}\right)$$

Obviously, $L > 0$ and $L(S^*, I^*) = 0$. The time derivative of L is not positive if $\mathcal{R}_0 > 1$:

$$\frac{dL}{dt} = -\mu(S - S^*)^2 - (\mu + \alpha)(I - I^*)^2$$

Therefore, the EE E^* is globally asymptotically stable in G if $\mathcal{R}_0 > 1$.

■

SIR and SIRS models

Certain infectious diseases provide lasting immunity, while others confer temporary acquired immunity. These two types of diseases can be represented by the SIR model for permanent immunity and the SIRS model for temporary acquired immunity. The total population N is divided into three compartments, where N equals the sum of individuals in the susceptible class (S), infectious individuals (I), and those who have recovered (R). The system of equations of model SIR [3]:

$$\begin{cases} \frac{dS}{dt} = \Lambda - \beta SI - \mu S + \gamma R \\ \frac{dI}{dt} = \beta SI - (\kappa + \mu + \alpha)I \\ \frac{dR}{dt} = \kappa I - (\mu + \gamma)R \end{cases}$$

In this context, the parameters Λ, μ, β, κ, and α are positive constants, while γ is a non-negative constant. We assume that κ denotes the rate at which infectives recover. When individuals who have recovered gain permanent immunity (γ = 0), we have the SIR model. Conversely, when individuals acquire temporary immunity (γ ≠ 0), it corresponds to the SIRS model.

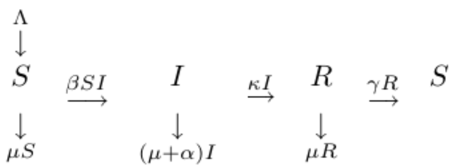

Figure 4: SIR and SIRS model

The disease-free equilibrium (DFE) $E^0 = (S^0, I^0, R^0) = \left(\frac{\Lambda}{\mu}, 0,0\right)$ and basic reproduction number $\mathcal{R}_0 = \frac{\Lambda\beta}{\mu(\phi + \mu + \alpha)}$. Feasible region is:

$G = \left\{(S, I, R) \in \mathbb{R}^3{}_+ :\ S \geq 0,\ \ I \geq 0, R \geq 0,\ S + I + R \leq \frac{\Lambda}{\mu}\right\}$ which is positively invariant with respect to SIR and SIRS model's system of equations.

Theorem 4: The DFE E^0 is globally asymptotically stable in G if

$\mathcal{R}_0 \leq 1$.

Proof: To prove it we can define $L: \{(S, I, R) \in G: S, I, R > 0\}$:

$$L(S, I, R) = \frac{1}{2}[(S - S^0) + I + R]^2 + \frac{\alpha + 2\mu}{\beta} I + \frac{\alpha + 2\mu}{2\kappa} R^2$$

Obviously, $L > 0$ and $L(S^0, I^0) = 0$. The time derivative of L is not positive if $\mathcal{R}_0 \leq 1$:

$$\frac{dL}{dt} = -\mu[(S - S^0) + R]^2 - (\mu + \alpha)I^2 - \frac{(\alpha + 2\mu)(\mu + \gamma)}{\kappa}R^2$$

$$-\frac{(\alpha + 2\mu)(\phi + \mu + \alpha)}{\beta}(1 - \mathcal{R}_0)I$$

Therefore, the DFE E^0 is globally asymptotically stable in G if $\mathcal{R}_0 \leq 1$.

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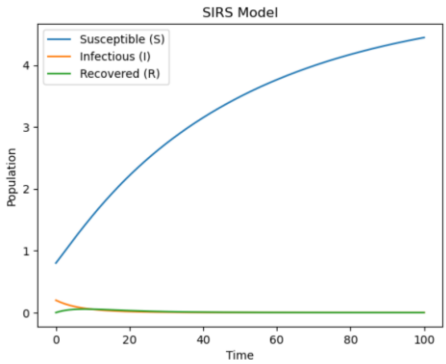

Figure 5: SIRS model implementation in Python when $\mathcal{R}_0 \leq 1$

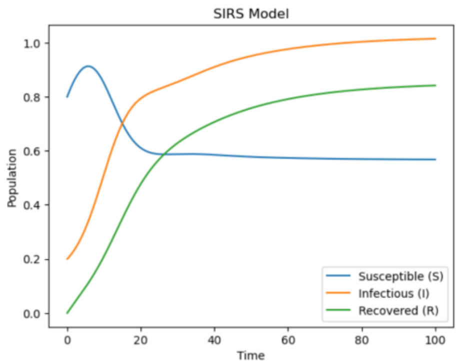

Figure 6: SIRS model implementation in Python when $\mathcal{R}_0 >$

Let's consider the endemic equilibrium (EE) of SIR and SIRS model:

$$E^* = (S^*, I^*, R^*) = \left(\frac{S^0}{R^0}, \frac{\mu(\mu+\gamma)(\kappa+\alpha+\mu)(\mathcal{R}_0-1)}{\beta(\kappa\mu+(\mu+\gamma)(\alpha+\mu))}, \frac{\kappa\mu(\kappa+\alpha+\mu)(\mathcal{R}_0-1)}{\beta(\kappa\mu+(\mu+\gamma)(\alpha+\mu))}\right).$$

Theorem 5: The EE E^* is globally asymptotically stable in G if $\mathcal{R}_0 > 1$.

Proof: To prove it we can define function using combination of composite quadratic Lyapunov function and logarithmic Lyapunov function,

$$L: \{(S, I, R) \in G: S, I, R > 0\}$$

$$L(S, I, R) = \frac{1}{2}[(S - S^*) + (I - I^*) + (R - R^*)]^2 + \frac{\alpha + 2\mu}{\beta}\left(I - I^* - I^* ln\frac{I}{I^*}\right)$$

$$+\frac{\alpha + 2\mu}{2\kappa}(R - R^*)^2$$

Obviously, $L > 0$ and $L(S^*, I^*, R^*) = 0$. The time derivative of L is not positive if $\mathcal{R}_0 > 1$:

$$\frac{dL}{dt} = -\mu[(S - S^*) + (R - R^*)]^2 - (\mu + \alpha)(I - I^*)^2$$

$$-\frac{(\alpha + 2\mu)(\mu + \gamma)}{\kappa}(R - R^*)^2$$

Therefore, the EE E^* is globally asymptotically stable in G if $\mathcal{R}_0 > 1$.

Место для уравнения.

Conclusion

In conclusion, even though at the moment there is no general method for finding the Lyapunov function, this review explored the diverse landscape of Lyapunov functions and their contributions to achieving global stability in dynamical systems. Examples of using Lyapunov functions for proving global stability of different epidemic models considered. A further step in this area could be constructing a general method of global stability looking at known types of Lyapunov functions. As we already mentioned it is extremely important for the whole of humanity.

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ЛЯПУНОВ ФУНКЦИЯЛАРЫНЫҢ ТҮРЛЕРІ ЖӘНЕ ЖАҺАНДЫҚ ТҰРАҚТЫЛЫҚ

Аңдатпа. Динамикалық жүйелердің тұрақтылығын талдау олардың мінез-құлқын түсінудің және әртүрлі қолданбаларда сенімді өнімділікті қамтамасыз етудің маңызды аспектілерінің бірі болып табылады. Тепе-теңдік нүктелерінің тұрақтылығын бағалаудың қуатты математикалық құралын ұсына отырып, Ляпунов функциялары бұл талдауда шешуші рөл атқарады. Қазіргі уақытта Ляпунов функциясын табудың жалпы әдісі жоқ екеніне қарамастан, бұл шолуда Ляпунов функцияларының әртүрлі ландшафты және олардың динамикалық жүйелердегі жаһандық тұрақтылыққа қол жеткізудегі үлесі зерттеледі. Тереңірек түсіну үшін эпидемиялық модельдердің мысалдары және олардың Ляпунов функцияларының Python тілінде жазылған графикалық іске асырулары әртүрлі репродукциялық сандармен қамтылған. Сонымен қатар

Ляпуновтың ғаламдық асимптотикалық тұрақтылық теоремасының тарихы қысқаша қарастырылған болатын. Осы саладағы ықтимал зерттеу идеялары артиклдің соңында берілген.
Түйін сөздер: Ляпунов функциялары, ғаламдық тұрақтылық, динамикалық жүйелер, SIR, SEIR, тұрақтылық

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ТИПЫ ФУНКЦИЙ ЛЯПУНОВА И ГЛОБАЛЬНАЯ УСТОЙЧИВОСТЬ

Аннотация. Анализ устойчивости динамических систем является одним из важнейших аспектов в понимании их поведения и обеспечения надежной работы в различных приложениях. Функции Ляпунова играют ключевую роль в этом анализе, предлагая мощный математический инструмент для оценки устойчивости точек равновесия. Несмотря на то, что на данный момент не существует общего метода нахождения функции Ляпунова, в этом обзоре исследуется разнообразный ландшафт функций Ляпунова и их вклад в достижение глобальной устойчивости в динамических системах. Для более глубокого понимания включены примеры моделей эпидемий и их функций Ляпунова с графическими реализациями на Python в разных числах воспроизводства. Кроме того, кратко рассмотрена история теоремы Ляпунова о глобальной асимптотической устойчивости. Возможные идеи дальнейших исследований в этой области приведены в конце.
Ключевые слова: Функции Ляпунова, глобальная устойчивость, динамические системы, SIR, SEIR, устойчивость.